15. L. P. Ul'yanov, V. M. Neimark, Yu. G. Yanovskii, and S. I. Sergeenkov, Zavod. Lab., 39, No. 11, 1402-1406 (1973).
16. Z. P. Shul'man, B. M. Khusid, B. P. Khizhinskii, et al., "Electrorheological vibration-proof insulation of oscillatory systems," Preprint No. 22, Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk (1986).

MOTION OF A HIGHLY VISCOUS NON-NEWTONIAN LIQUID IN RESERVOIRS

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Self-similar solutions are found for the equation of spreading of a thin layer of high viscosity non-Newtonian liquid in the presence of a constant power source. Results are compared with experimental data.

It was shown in [1] that in the approximation of a geometrically thin layer ($h_0/L \ll 1$, where h_0 is the layer height) flow over a horizontal plane of a layer of high viscosity rheologically complex liquid $Re_h = (\rho g h/\eta)(h/L)^2 \ll 1$ can be described by using an equation for the change in layer height [$h = h(x, y, t)$] with time:

$$\frac{\partial h}{\partial t} = \nabla\left(\frac{\rho g h^3}{\eta_0}\beta\nabla h\right),\ \beta = \int_0^1 (1-\xi)^2\Psi\left[\frac{\rho g h}{\tau_0}|\nabla h|(1-\xi)\right]d\xi. \quad (1)$$

For a Newtonian liquid $\Psi = 1$, $\beta = 1/3$, for a power model ($\tau = k\gamma^n$)

$$\Psi = \left(\frac{\tau}{\tau_0}\right)^{\frac{1}{n}-1},\ \beta = \frac{n}{2n+1}\left(\frac{\rho g h}{\tau_0}|\nabla h|\right)^{\frac{1}{n}-1},$$

where τ_0 is the value of the shear stress at which the viscosity is equal to η_0.

In a radial coordinate system Eq. (1) can be written in the form

$$\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\rho g h^3}{\eta_0}\beta\frac{\partial h}{\partial r}\right]. \quad (2)$$

For a power model, in particular, Newtonian, at n = 1 Eq. (2) has a self-similar solution describing spreading of a liquid with a constant supply at flow rate Q (Fig. 1). This solution can be found as in the analogous filtration problem [2]. The dependence of the height h on radius r and time t can be expressed in terms of self-similar dimensionless variables length ξ and time ζ in the following manner:

$$h = \frac{h_0 f(\xi)}{\Phi(\zeta)},\ \xi = \frac{r}{h_0\varphi(\zeta)},\ \zeta = \frac{t}{t_0}, \quad (3)$$

where $\Phi(\zeta)$, $\varphi(\zeta)$, $f(\xi)$ are some functions. Equation (2) must be solved simultaneously with the condition of linear increase over time of the liquid volume:

$$Qt = \int_0^\infty 2\pi h r dr. \quad (4)$$

Substitution of Eq. (3) in Eq. (4) yields

$$Qt_0\zeta = 2\pi h_0^3\frac{\varphi^2(\zeta)}{\Phi(\zeta)},\ \int_0^\infty f(\xi)\xi d\xi = 1. \quad (5)$$

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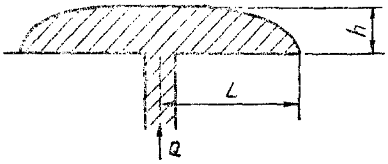

Fig. 1. Diagram of liquid spreading.

We transform Eq. (2) into an expression for the function f(ξ) by substituting Eq. (3) therein. To do this we use the expressions

$$\frac{\partial h}{\partial t} = -\frac{1}{t_0}\frac{h_0}{\Phi(\zeta)}\left[\frac{f(\xi)}{\Phi(\zeta)}\frac{d\Phi}{d\zeta} + \frac{1}{\varphi(\zeta)}\frac{d\varphi}{d\zeta}\xi\frac{df}{d\xi}\right], \tag{6}$$

$$\frac{\partial h}{\partial r} = \frac{1}{\Phi(\zeta)\varphi(\zeta)}\frac{df}{d\xi}. \tag{7}$$

For the power model

$$\beta = \frac{n}{2n+1}\left[\frac{\rho g h_0}{\tau_0}\frac{f(\xi)}{\Phi(\zeta)}\left|\frac{1}{\Phi(\zeta)\varphi(\zeta)}\frac{df}{d\xi}\right|\right]^{\frac{1}{n}-1}. \tag{8}$$

Substituting Eqs. (6)-(8) in Eq. (2) and equating the coefficients of ζ, we obtain

$$\frac{1}{\Phi(\zeta)}\frac{d\Phi(\zeta)}{d\zeta} = C\frac{1}{\varphi(\zeta)}\frac{d\varphi(\zeta)}{d\zeta},$$

where C is an arbitrary constant. This condition is satisfied at $\Phi = \varphi^C$. With consideration of the first expression of Eq. (5) we find: $Qt_0\zeta = 2\pi h_0^3\varphi^{2-C}$. Hence

$$A = \frac{Qt_0}{2\pi h_0^3},\ \varphi = (A\zeta)^m,\ \Phi = (A\zeta)^{2m-1},\ C = \frac{2m-1}{m}. \tag{9}$$

The parameters in these expressions are defined after substitution of Eq. (9) into Eq. (2):

$$m = \frac{2(n+1)}{5+3n},\ t_0 = A^4\left(\frac{k}{\rho g h_0}\right)^{\frac{1}{n}},\ \frac{Q}{2\pi h_0^3}\left(\frac{k}{\rho g h_0}\right)^{\frac{1}{n}}A^3 = 1, \tag{10}$$

Choosing A = 1, we find an expression for the layer height scale

$$h_0 = \left[\frac{Q}{2\pi}\left(\frac{k}{\rho g}\right)^{\frac{1}{n}}\right]^{\frac{n}{3n+1}}. \tag{11}$$

The equation for the function f(ξ) can be written in the form

$$-\left[(2m+1)f(\xi) + m\xi\frac{df}{d\xi}\right] = \frac{1}{\xi}\frac{d}{d\xi}\left[\xi f^3\left(f\left|\frac{df}{d\xi}\right|\right)^{\frac{1-n}{n}}\frac{df}{d\xi}\right]. \tag{12}$$

It must be solved simultaneously with the integral expression of Eq. (5)

$$\int_0^\infty f(\xi)\xi d\xi = 1. \tag{13}$$

Even for n = 1 Eqs. (12), (13) cannot be solved analytically. It follows from the self-similar solution that:

$$h = \frac{h_0 f(\xi)}{\zeta^{2m-1}},\ \xi = \frac{r}{h_0\zeta^m},\ \zeta = \frac{t}{t_0},\ t_0 = \left(\frac{k}{\rho g h_0}\right)^{\frac{1}{n}}.$$

For a Newtonian liquid, we find from Eqs. (10), (11): $m = 1/2$, $h_0 = (Q\eta/2\pi\rho g)^{1/4}$. The self-similar expressions show that the layer spreads by a power law (constant ξ level):

$$R(t) = \lambda\left(\frac{Q}{2\pi}\right)^{\frac{2+n}{3n+5}}\left(\frac{\rho g}{k}\right)^{\frac{1}{3n+5}}t^{\frac{2(n+1)}{3n+5}}. \tag{14}$$

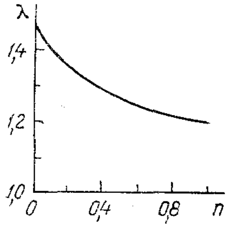

Fig. 2

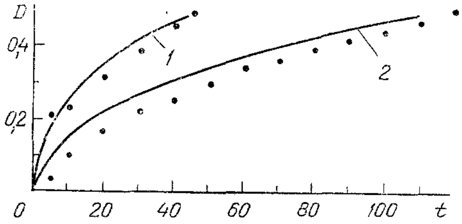

Fig. 3

Fig. 2. Coefficient λ vs. n.

Fig. 3. Comparison of self-similar solution (curves) with experimental data (points) for flow rates $Q = 27.9 \cdot 10^{-5}$ (1) and $9 \cdot 10^{-5}$ m³/sec (2).

The dimensionless quantity λ depends on the rheological parameter n.

The approximate form of the layer and the value of λ can be found by using the quasi-steady state solution of Eq. (2):

$$\frac{d}{dr}\left(r\frac{\rho g h^3}{\eta_0}\beta\frac{dh}{dr}\right) = 0. \quad (15)$$

We add to Eq. (15) the boundary condition

$$h|_{r=R(t)} = 0, \quad (16)$$

where R(t) is the radius of the liquid layer. Equations (15), (16) can be integrated easily:

$$h = \bar{h}(\bar{r})\tilde{h}_0, \quad \bar{r} = \frac{r}{R(t)}. \quad (17)$$

Here

$$\tilde{h}_0 = \left[\left(\frac{Q}{2\pi}\right)^n \frac{R^{1-n}(t)\,k}{\rho g}\left(\frac{2n+1}{n}\right)^n\right]^{\frac{1}{2(n+1)}},$$

$$\begin{cases} \bar{h}(\bar{r}) = \left[\frac{2(n+1)}{1-n}(1-\bar{r}^{1-n})\right]^{\frac{1}{2(n+1)}}, & n \neq 1; \\ \bar{h}(\bar{r}) = \left[4\ln\frac{1}{\bar{r}}\right]^{\frac{1}{4}}, & n = 1. \end{cases}$$

Only for n < 1 is a finite $\tilde{h}$ value realized at the center. Applying Eq. (17), we find the liquid volume in the layer [3]:

$$V = 2\pi R^2(t)\,\tilde{h}_0 I(n), \quad I(n) = \int_0^1 \bar{r}\,\bar{h}\,d\bar{r}, \quad (18)$$

where

$$I(n) = \begin{cases} \Gamma\left(\frac{5}{4}\right)\Big/ 2^{3/4}, & n = 1; \\ \frac{2(n+1)}{1-n}\,\dfrac{\Gamma\left(\frac{2}{1-n}\right)\Gamma\left(1+\frac{1}{2(n+1)}\right)}{\Gamma\left(1+\frac{2}{1-n}+\frac{1}{2(n+1)}\right)}, & n \neq 1. \end{cases}$$

Substituting in Eq. (18) the expression for $\tilde{h}_0$ we find that the relationship V = Qt is satisfied, when R(t) is defined by Eq. (14) at

$$\lambda(n) = \left[\frac{1}{I(n)}\right]^{\frac{2(n+1)}{3n+5}} \left(\frac{n}{2n+1}\right)^{\frac{n}{3n+5}}. \qquad (19)$$

Figure 2 shows values of λ for various n, calculated by Eq. (19).

The slow axisymmetric flow of a Newtonian liquid over a horizontal plane with constant flow rate supply was studied in [4]. It was shown that for large times the dependence of layer radius on time is given by a law $R(t) = a\, Q^{3/8}(\rho g/k)^{1/8}(\rho g/k)^{1/8}t^{1/2}$. Numerically the value $a = 0.62$ was determined [4]. The expression of [4] can be obtained from Eq. (14) at n = 1 for $\lambda(I) = (2\pi)^{3/8}a$. The experiments in [4] were performed for water-glycerine mixtures with kinematic viscosity of (0.05-9.15) cm^2/sec. The asymptotic expression is applicable 10-30 sec after commencement of spreading. For the experimental value $a = 0.65$ [4] λ(I) = 1.29. Calculation with Eq. (19) gives λ(I) = 1.19.

A comparison with experimental data for spreading of a layer of a solution containing 8% polyisobutylene by mass is shown in Fig. 3 for two constant flow rates. The dependence of stress τ on shear velocity γ over the entire γ range is described by the equation $\tau = 351.3\gamma^{0.458}$. It is evident that good agreement has been achieved between calculated and experimental values.

NOTATION

h, L, layer thickness and height, m; Re, Reynolds number; u, spreading velocity, m/sec; ρ, η, liquid density and viscosity, kg/m^3 and Pa·sec; τ, stress, Pa; γ, shear velocity, 1/sec; t, time, sec; x, y, r, θ, coordinates; n, exponent in power law; k, consistency coefficient, $Pa\cdot sec^{-n}$; Q, flow rate, m^3/sec; t_0, time scale; λ, correction coefficient; D, layer diameter, m.

LITERATURE CITED

1. Z. P. Shul'man and B. M. Khusid, Nonsteady State Convective Transport Processes in Hereditary Media [in Russian], Minsk (1983).
2. G. I. Barenblatt, V. M. Entov, and V. M. Ryzhik, Theory of Nonsteady State Liquid and Gas Filtration [in Russian], Moscow (1972).
3. V. V. Mel'nichenko, O. I. Mikulin, B. M. Khusid, et al., Vests. Akad. Navuk Beloruss. SSSR, Ser. Fiz.-Energ. Navuk, No. 2, 83-87 (1988).
4. A. G. Zatsepin, A. G. Kostyanoi, and G. I. Shapiro, Dokl. Akad. Nauk SSSR, 265, No. 1, 193-195 (1982).